Categorical Connections between Compact Riemann Surfaces and Complex Field Extensions

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1. Introduction

Much of the work we did early in the semester was concerned with proving the existence of meromorphic functions on a given Riemann surface. One important consequence of this work is the following result: given a compact Riemann surface X of genus g and a point $p \in X$, there exists a nonconstant meromorphic function $f : X \to \mathbb{CP}^1$ with pole of order at most $g+1$ at p. This theorem allows us to associate to each compact connected Riemann surface Σ a piece of algebraic data, namely the field of meromorphic functions on Σ, which we denote using k_Σ. In fact, this association can be strengthened into an equivalence of categories, which we make precise in the following.

Theorem 1.1 (The Main Theorem)**.** *Let $\mathcal{C}$ denote the category whose objects are compact connected Riemann surfaces and whose morphisms are holomorphic maps, and let $\mathcal{D}$ denote the category of finite algebraic field extensions of $\mathbb{C}(z)$ with field inclusions as its morphisms. Then $\mathcal{C}$ and $\mathcal{D}$ are (dually) equivalent categories.*

While not immediately obvious, there are several advantages to this more categorical approach. Of interest, in particular, are the several algebraic and number-theoretic consequences of this theorem, one of which we state (and prove) immediately.

Theorem 1.2 (Lüroth's Theorem)**.** *Let K be a field such that $\mathbb{C} \subset K \subset \mathbb{C}(z)$, and assume $K \neq \mathbb{C}$. Then K and $\mathbb{C}(z)$ are isomorphic.*

To prove Theorem 1.2, we'll need the following result, which we proved in Problem Set 3.

Lemma 1.3. *Let X and Y be compact connected Riemann surfaces, and let $f : X \to Y$ be a holomorphic map of degree d. Then k_Y includes into k_X by precomposing with f, and k_X is an algebraic extension of k_Y of degree d.*

Proof of Lüroth's Theorem. Begin by recalling that $\mathbb{C}(z)$ is the field of meromorphic functions on $\mathbb{CP}^1$. Note that the assumption $K \neq \mathbb{C}$ and the fact that $\mathbb{C}$ is algebraically closed together imply that K has nonzero transcendence degree as a field extension of $\mathbb{C}$. The inclusion $K \subset \mathbb{C}(z)$ tells us that $\operatorname{trdeg}_{\mathbb{C}}(K) = 1$. Thus, we may apply Theorem 1.1 to $K \subset \mathbb{C}(z)$; it follows that there is a holomorphic map $f : \mathbb{CP}^1 \to \Sigma_K$, where Σ_K is the compact connected Riemann surface associated to the field K. Now, the existence of a nonconstant meromorphic function on Σ_K yields a holomorphic $h : \mathbb{CP}^1 \to \mathbb{CP}^1$ given by postcomposing this meromorphic function with f. By Lemma 1.3, the degree of h is the degree of the resulting field extension, i.e., $\deg(h) = [\mathbb{C}(z) : \mathbb{C}(z)] = 1$. Now, $\deg(h) = 1$ forces $\deg(f) = 1$, since h is given by precomposing the meromorphic function on Σ_K with f. It follows that f is a biholomorphism, and therefore Theorem 1.1 forces $K \cong \mathbb{C}(z)$. □

Theorem 1.2 is just one of the many applications of Theorem 1.1, the rest of which are discussed in Section 3. Section 2 is dedicated to illustrating Theorem 1.1.[1] Henceforth, when we refer to fields, we mean field extensions of $\mathbb{C}$ of transcendence degree 1, unless specified otherwise. Since our discussion of Riemann surfaces is solely concerned with those that are both compact and connected, implicitly assume that such adjectives accompany the phrase "Riemann surface" from here on out. What follows is essentially an amalgamation of parts of Chapters 4 and 11 of [1].

[1] Despite my best attempts, I was unable to come up with or find a sufficiently rigorous proof of Theorem 1.1 without referring to more high-powered tools from algebraic geometry. While I will describe a general procedure for proving Theorem 1.1, this will be at best a sketch of the proof.

2. The Main Theorem

2.1. **Illustrating the Theorem.** As mentioned in Section 1, to each Riemann surface Σ we may associate its field of meromorphic functions k_Σ. Moreover, given a holomorphic map $f : \Sigma_1 \to \Sigma_2$, we get a map of of fields $k_{\Sigma_2} \to k_{\Sigma_1}$ (which, recall, is necessarily an inclusion) given by pulling a meromorphic function on Σ_2 back to a meromorphic function on Σ_1 by precomposing with f. Denote this map of fields by f^*.

For the other direction, given a field K, we would like to associate a Riemann surface X to K such that the field of meromorphic functions on X is isomorphic to K.

Theorem 2.1. *Let K be any finite extension of $\mathbb{C}(z)$. Then there is a compact, connected Riemann surface Σ such that $k_\Sigma = K$. In particular, the data of a field extension $K/\mathbb{C}(z)$ gives us a Riemann surface Σ and a holomorphic $f : \Sigma \to \mathbb{CP}^1$.*

Proof. Begin by noting that because K is a finite extension of $\mathbb{C}(z)$, we may write K as

$$\mathbb{C}(z)[w]/(P),$$

where P is some irreducible polynomial in w with coefficients in $\mathbb{C}(z)$. We claim that, without loss of generality, we may assume $P \in \mathbb{C}[z, w]$. Clearing denominators, we get $cP \in \mathbb{C}[z, w]$ for some $c \in \mathbb{C}[z]$, and we may assume that the coefficients in cP do not share a common factor (i.e., cP is primitive). Since P is irreducible in $\mathbb{C}(z)[w]$, cP remains irreducible in $\mathbb{C}(z)[w]$, since we are multiplying by a unit. Moreover, cP is primitive in $\mathbb{C}[z][w]$, implying that cP is irreducible in $\mathbb{C}[z][w]$ by Gauss' Lemma. Thus, assume $P \in \mathbb{C}[z, w]$. Since P is irreducible, the ideal it generates in $\mathbb{C}[z, w]$ is prime, and we have that $\mathbb{C}[z, w]/(P)$ is an integral domain. We see that K must be the field of fractions of $\mathbb{C}[z, w]/(P)$; one way to see this is by recalling the fact that localization respects quotients.

Now, let P_1 and P_2 be relatively prime polynomials in $\mathbb{C}[z, w]$. When viewed as elements of $\mathbb{C}(z)[w]$, both polynomials must still be coprime by Gauss' Lemma, implying that we may write $\lambda P_1 + \mu P_2 = 1$ for $\lambda, \mu \in \mathbb{C}(z)[w]$. Clear denominators to give $\rho(z) = \tilde{\lambda} P_1 + \tilde{\mu} P_2$ for some $0 \neq \rho \in \mathbb{C}[z]$. It is not hard to see that the set of all ρ which can be expressed in this way forms an ideal of $\mathbb{C}[z]$: if ρ_1, ρ_2 are in this set, then we can write

$$\rho_1(z) + \rho_2(z) = (\tilde{\lambda}_1 + \tilde{\lambda}_2)P_1 + (\tilde{\mu}_1 + \tilde{\mu}_2)P_2,$$

and

$$q(z)\rho(z) = q\tilde{\lambda}P_1 + q\tilde{\mu}P_2$$

for all $q \in \mathbb{C}[z]$. Therefore, because $\mathbb{C}[z]$ is a principal ideal domain, the ideal in question is generated by some (monic) element $\rho_0(z)$ we call *the resultant* of P_1 and P_2 (requiring the resultant to be monic eliminates any sort of ambiguity about which generator to pick).

By definition, we see that the projection of the zero locus of $P_1(z, w)$ and $P_2(z, w)$ to the z-coordinate is contained in the vanishing set of $\rho_0(z)$. In other words, if (a, b) is a common root of P_1 and P_2, then a is a root of ρ_0. There are only finitely many possibilities for a; a symmetry argument implies that there are only finitely many possibilities for b. Therefore, P_1 and P_2 have only finitely many common roots.

Now, recall that we have some irreducible polynomial $P \in \mathbb{C}[z, w]$. Let $X \subset \mathbb{C}^2$ denote its zero locus, and let S denote the set of singular points in X—the points in X where both of the partials P_z and P_w vanish. Applying the above result to P and P_w tells us that S must be finite. We've seen before that a smooth, complex algebraic curve is a Riemann surface, so $X \setminus S$ is indeed a Riemann surface (we're throwing out the singular points). Let $\pi : X \to \mathbb{C}$ denote projection onto the z-coordinate. View P as a polynomial in w with coefficients in $\mathbb{C}[z]$, and let F be the finite set of roots of the leading coefficient of P regarded as a polynomial in w. Define $S^+ = \pi^{-1}(\pi(S) \cup F) \subset X$. It is not hard to see that S^+ is itself finite: If $(z_0, w_0) \in S^+$, then $z_0 = \pi(z_0, w_0) \in \pi(S) \cup F$ (which is a finite set). We also know $P(z_0, w_0) = 0$, implying that for each z_0 there are only finitely many possible w_0 such that $(z_0, w_0) \in S^+$, unless $P(z_0, w)$ is identically 0. This would force $(z - z_0)|P$, contradicting irreducibility (ignore the trivial case where $P = z - z_0$).

Letting $E = \pi(S) \cup F \cup \{\infty\}$, viewed as a subset of $\mathbb{CP}^1$, we see that π is a proper holomorphic map

$$\pi : X \setminus S^+ \to \mathbb{CP}^1 \setminus E.$$

Let Δ denote the set of critical values of π. The proper branched covering π gives a monodromy $\rho : \pi_1(\mathbb{CP}^1 \setminus (E \cup \Delta)) \to S_d$ from which we can recover $X \setminus S^+$ by Riemann's Existence Theorem. Recall from early in the semester that this data also gives us a compact Riemann surface X^* containing $X \setminus S^+$ as a dense open subset along with a holomorphic map from X^* to $\mathbb{CP}^1$ given by "putting back the punctures" S^+.

Thus, to each irreducible polynomial P, we may associate a compact Riemann surface $\Sigma = X^*$ equipped with a holomorphic map to $\mathbb{CP}^1$ with degree $\deg_w(P)$. It remains to show that Σ is connected. To do so, we will show that $Z(P) \subset \mathbb{C}^2$ is connected. Suppose for a contradiction that $Z(P)$ has two components, so that Σ is a disjoint union of compact Riemann surfaces Σ_1 and Σ_2. By considering k_{Σ_i}, we see that Σ_i can be associated to a polynomial $Q_i \in \mathbb{C}(z)[w]$ as the compactification of its vanishing locus. However, this implies that Σ is associated to the polynomial Q_1Q_2, which forces $P = Q_1Q_2$, contradicting the irreducibility of P. In particular, we have P is reducible in $\mathbb{C}(z)[w]$; Gauss' Lemma will tell us that P is reducible in $\mathbb{C}[z, w]$. Hence, Σ is connected.

Finally, we claim that $k_\Sigma = K$. Because Σ comes equipped with a degree d holomorphic map $f : \Sigma \to \mathbb{CP}^1$, we see that $[k_\Sigma : \mathbb{CP}^1] = d$ by Lemma 1.3. But we also have a natural inclusion $K \subset k_\Sigma$, implying that $d = [k_\Sigma : \mathbb{CP}^1] = [k_\Sigma : K][K : \mathbb{CP}^1]$. Since K is a degree-d extension of $\mathbb{C}(z)$, it follows that $[k_\Sigma : K] = 1$, forcing the equality $k_\Sigma = K$. □

2.2. **An Outline of the Proof.** In this section, rather than focusing on proofs, we instead give a sketch of the steps needed to prove Theorem 1.1. Let's begin by setting things up more categorically: we have a contravariant functor $\Phi : \mathcal{C} \to \mathcal{D}$ taking $\Sigma \mapsto k_\Sigma$ and f to the aforementioned map of fields f^*. Checking that Φ is indeed a contravariant functor is straightforward.

Now, we need to define a functor $\Psi : \mathcal{D} \to \mathcal{C}$ such that $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are naturally isomorphic to $\mathrm{id}_\mathcal{C}$ and $\mathrm{id}_\mathcal{D}$, respectively. Theorem 2.1 tells us how to associate to each field a Riemann surface Σ and a holomorphic map $\Sigma \to \mathbb{CP}^1$. However, there are several drawbacks to this approach, especially from a categorical perspective:

(1) Our construction of Σ from K is ad hoc and noncanonical.

(2) This construction also obscures what Σ really is as a Riemann surface. We see that it is the compactification of some algebraic curve in $\mathbb{C}^2$, however this construction can be difficult to work with (especially near the singular points).

(3) Given an inclusion of fields $L \subset K$, let Σ_L and Σ_K denote the associated Riemann surfaces given by Theorem 2.1. It is not at all obvious from the approach taken in Theorem 2.1 that there should be a canonical holomorphic map from $\Sigma_K \to \Sigma_L$.

Hence, the following definition comes into play:

Definition 2.2. Let K be any field. A *valuation* on K is a surjective map $\nu : K \to \mathbb{R} \cup \{\infty\}$ with the following significance:

(1) $\nu^{-1}(\infty) = \{0\}$;

(2) $\nu(a+b) \geq \min(\nu(a), \nu(b))$ for all $a, b \in K$;

(3) $\nu(ab) = \nu(a) + \nu(b)$ for all $a, b \in K$.[2]

We say that two valuations ν_1 and ν_2 are *equivalent* if $\nu_1 = c\nu_2$ for some $c \in \mathbb{R}^+$ (also note that $c\nu$ is a valuation). A valuation is said to be *trivial* if it takes all nonzero elements of K to 0. If K is a subfield of some larger field L, then clearly any valuation on L restricts to one on K. If the restriction of such a valuation ν to K is trivial, we say that v is a valuation of L *over* K.

Given some valuation ν, let R_ν be the subset of K consisting of those elements with nonnegative valuations; let $I_\nu \subset R_\nu$ be those elements with strictly positive valuations. In fact, R_ν is a ring; we call it the *valuation ring*. It is not hard to see that I_ν is a maximal ideal in R_ν, so R_ν/I_ν is a field, which we call the *residue class field* of ν.[3]

Let K be any field in $\mathcal{D}$. Let $\mathrm{Val}(K)$ denote the set of equivalence classes of valuations on K over $\mathbb{C}$. We endow $\mathrm{Val}(K)$ with a Riemann surface structure; with this structure $\mathrm{Val}(K)$ is an example of a *Zariski-Riemann space*.[4] Despite the abstractness of this approach, we will see immediately that it is preferable to the one taken in Theorem 2.1. In particular, if $L \subset K$ there is a natural map from $\mathrm{Val}(K) \to \mathrm{Val}(L)$ given by considering the restriction of a valuation on K to a valuation on L. Moreover, if we start with a Riemann surface Σ, there is a bijective correspondence between the points of Σ and valuations in $\mathrm{Val}(k_\Sigma)$. The Riemann surface structure we give to $\mathrm{Val}(K)$ will guarantee that all of these maps are holomorphic (in particular the map between Σ and $\mathrm{Val}(k_\Sigma)$ is a holomorphic isomorphism).

Therefore, we can let $\Psi : \mathcal{D} \to \mathcal{C}$ be the functor taking a field K to the Riemann surface $\mathrm{Val}(K)$ and a morphism of fields $L \hookrightarrow K$ to a holomorphic map between the Riemann surfaces $\mathrm{Val}(K) \to \mathrm{Val}(L)$ given by restriction. We see that

$$\Psi(\Phi(\Sigma)) = \Psi(k_\Sigma) = \mathrm{Val}(k_\Sigma) \cong \Sigma$$

and that

$$\Phi(\Psi(K)) = \Phi(\mathrm{Val}(K)) = k_{\mathrm{Val}(K)}.$$

Now, it is not difficult to check that, for any field K and $\nu \in \mathrm{Val}(K)$, the residue class field R_ν/I_ν is isomorphic to $\mathbb{C}$. This allows us to, for each $f \in K$, define a map $e_f : \mathrm{Val}(K) \to \mathbb{CP}^1$ given by

$$e_f(\nu) = \begin{cases} \infty & \text{if } \nu(f) < 0; \\ \text{the value of } [f] \in R_\nu/I_\nu \cong \mathbb{C} & \text{otherwise.} \end{cases}$$

Each of these maps will be holomorphic (in fact, the Riemann surface structure on $\mathrm{Val}(K)$ will be characterized by this property), implying that there is an inclusion of fields $K \subset k_{\mathrm{Val}(K)}$. Finally, if K is a degree-d field extension of $\mathbb{C}(z)$, we will also need to prove the existence of a degree d branched cover $\mathrm{Val}(K) \to \mathbb{CP}^1$. This will tell us that $[k_{\mathrm{Val}(K)} : \mathbb{CP}^1] = d$, and the inclusion $K \subset k_{\mathrm{Val}(K)}$ (given

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[2] This definition can be generalized by replacing $\mathbb{R} \cup \{\infty\}$ by any totally ordered group G with ∞ adjoined.

[3] Suppose J is some ideal of R_v strictly containing I_ν. Then there is an element $a \in J$ such that $\nu(a) = 0$. It follows that $\nu(a^{-1}) = 0$, so $a^{-1} \in R_\nu$. Therefore, $1 = a^{-1}a \in J$.

[4] This is really a special case of a more general construction, in which the set of *valuation subrings* of a field extension F/k is endowed with the *Zariski topology*; the resulting space is called the *Zariski-Riemann space* of F/k. One can prove that Zariski-Riemann spaces are *locally ringed spaces*, and in the special case where F is the field of rational functions of a curve over an algebraically closed field k, we see that the corresponding Zariski-Riemann space is actually a *scheme*. However, we will avoid taking such an algebro-geometric approach in this paper; one can read more in [3]. Finally, valuation subrings have an explicit algebraic definition, but each valuation ring arises from a valuation on F. While we could avoid talking about valuations whatsoever, valuations allow us to associate arithmetic data to points, ideals, rings, etc., and therefore make the number-theoretic consequences more immediately apparent.

by the e_f's) then forces $K = k_{\text{Val}(K)}$ (K is a degree d extension of $\mathbb{C}(z)$). Therefore, $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are both naturally isomorphic to the identity, implying the desired equivalence of categories between $\mathcal{C}$ and $\mathcal{D}$.

It remains to discuss how $\text{Val}(K)$ is even topologized, let alone given a Riemann surface structure. To see how this is done, recall the maps $e_f : \text{Val}(K) \to \mathbb{CP}^1$ defined above. Endow $\text{Val}(K)$ with the coarsest topology such that this collection of maps is continuous. In other words, for each open $U \subset \mathbb{CP}^1$, declare $e_f^{-1}(U)$ to be open for all f, and let the topology on $\text{Val}(K)$ be the coarsest topology such that this is true. Now, in order to make $\text{Val}(K)$ into a Riemann surface, we need to provide it with an atlas of holomorphic charts. Again, we can use our maps e_f to do this. We have such an atlas on $\mathbb{CP}^1$, and we can pull back a chart (U, φ) in this atlas to an open set $e_f^{-1}(U)$ of $\text{Val}(K)$. Then, by restricting to some small-enough open subset of $e_f^{-1}(U)$, we get a chart on $\text{Val}(K)$. These charts will all be compatible—the transition maps are inherited from $\mathbb{CP}^1$ and are therefore surely holomorphic—, so we have an atlas of holomorphic charts, as desired.

Finally, we describe the aforementioned bijection between a Riemann surface Σ and $\text{Val}(\Sigma) := \text{Val}(k_\Sigma)$. To each point $p \in \Sigma$, associate the valuation ν_p given by $\nu_p(f) = \text{ord}_p(f)$, the order of vanishing of $f \in k_\Sigma$ at p. In other words, in a local chart of p, we write f as a Laurent series centered at $z(p) = 0$. Then $\nu_p(f)$ is defined to be the smallest nonvanishing power of z in this Laurent series representation of f. From this point of view, the valuation ring R is the ring of meromorphic functions that are holomorphic at p, and I is the set of meromorphic functions that vanish at p. It turns out that every valuation $\nu \in \text{Val}(\Sigma)$ is equivalent to ν_p for some p; this establishes the aforementioned bijection between Σ and $\text{Val}(\Sigma)$.

3. Consequences

3.1. **Connections to Algebraic Geometry.**

As noted previously, for an arbitrary field extension F/k, one can generalize our work in Section 2 and construct the *Zariski-Riemann space* of F/k, which is a locally ringed space and of independent algebro-geometric interest. One can read further about this generalization in [3]. However, there is a more immediate connection to algebraic geometry that we have essentially already seen in the case $k = \mathbb{C}$:

Theorem 3.1. *Let k be a field, and let $\mathcal{E}$ denote the category of* projective, nonsingular algebraic curves *and nonconstant morphisms between them. Then $\mathcal{E}$ is equivalent to the category of field extensions K/k with transcendence degree 1.*

Proving Theorem 3.1 as stated requires enough algebraic geometry to justify omitting the proof. Note that in the case where $k = \mathbb{C}$, Theorem 3.1 implies that $\mathcal{E}$ and $\mathcal{D}$ (and hence also $\mathcal{C}$) are equivalent as categories. Alternatively, by recalling some results we proved earlier this semester, we can sketch a proof of this special case by illustrating an equivalence between $\mathcal{C}$ and $\mathcal{E}$. Early on in the semester, we showed that every nonsingular, complex algebraic curve is a Riemann surface. In fact, we showed how to associate a Riemann surface to every complex algebraic curve—the normalization of the curve. Recall that we used this construction in our proof of Theorem 2.1. Later, we used Riemann-Roch to prove Chow's theorem:

Theorem 3.2 (Chow's theorem)**.** *A compact, connected, smooth complex-analytic curve $C \subset \mathbb{CP}^N$ is an algebraic variety. In particular, every compact Riemann surface is biholomorphic to some complex algebraic curve.*

Moreover, it turns out that every holomorphic map between two compact algebraic varieties is regular (i.e., a morphism of varieties); the canonical reference for this result is [4]. This result, in conjunction with Theorem 3.2 illustrates the way in which $\mathcal{C}$ and $\mathcal{E}$ are equivalent as categories.

3.2. **Algebraic Consequences.** Recall the following from earlier this semester: given a nonconstant holomorphic map of degree d $f : X \to Y$ of Riemann surfaces, we have a monodromy $\rho : \pi_1(Y \setminus \Delta) \to S_d$, where Δ is a discrete subset of Y. The image of this map is called the *monodromy group* of f.

Theorem 3.3. *Let $f : X \to Y$ be a nonconstant holomorphic map of Riemann surfaces. Then the Galois group of the corresponding field extension $k_Y \subset k_X$, $\mathrm{Gal}(k_X/k_Y)$, is isomorphic to the monodromy group of f.*

3.3. **Connections to Algebraic Number Theory.** There is also a beautiful connection between the theory of Riemann surfaces and that of algebraic number theory. To see this, begin by considering the rings $\mathbb{Z}$ and $\mathbb{C}[z]$. These rings are both Euclidean domains, and $\mathbb{Q}$ is analogous to $\mathbb{C}(z)$ (both are fields of fractions of the rings $\mathbb{Z}$ and $\mathbb{C}[z]$, respectively). Just as we considered finite extensions of $\mathbb{C}(z)$, we consider finite extensions of $\mathbb{Q}$, i.e., algebraic number fields. Denote such an extension by k. Recall that in the case where K is an algebraic extension of $\mathbb{C}(z)$, every valuation on K "lies over" a valuation ν_{z_0} corresponding to a point $z_0 \in \mathbb{C}(z)$. In the case that $k = \mathbb{Q}$, the valuations correspond precisely to the primes in $\mathbb{Z}$, and thus we see that for a general k, a valuation on k lies over the valuation corresponding to a prime p. Given any prime $p \in \mathbb{Z}$, we can write any rational number x as $p^v \frac{q}{r}$, where p does not divide either of the integers q or r. This gives us a valuation ν_p on $\mathbb{Q}$ given by $\nu_p(x) = v$. Checking that ν_p is indeed a valuation is straightforward, but it is a nontrivial fact (which we neglect to prove for the sake of brevity) that any valuation on $\mathbb{Q}$ is equivalent to one of the ν_p's. Recall that the valuation ν_{z_0} corresponding to $z_0 \in \mathbb{C}$ is constructed in exactly the same way as ν_p, with $(z - z_0)$ taking the place of the prime p. The following table (adapted from [2]) illustrates this analogy more succinctly:

Riemann Surfaces	Number Fields
$K = k_{\mathbb{CP}^1} = \mathbb{C}(z)$	$K = \mathbb{Q}$
$\mathcal{O}_X = \mathbb{C}[z]$	$\mathcal{O}_K = \mathbb{Z}$
$p \in \mathbb{C}$	$p\mathbb{Z}$
order of vanishing of $f(z)$ at p, $f \in \mathbb{C}(z)$	power of p dividing $n \in \mathbb{Q}$
germs of functions $\mathcal{O}_p = \{\sum a_n z_p^n\}$	$\mathbb{Z}_p = \{\sum a_n p^n\}$
residue field $\mathbb{C}$	residue field $\mathbb{F}_p$
finite extension $k_X/\mathbb{C}(z)$	algebraic extension $L/\mathbb{Q}$

While this analogy is interesting and useful, it is important to also recognize the ways in which the two situations are not analogous. In particular, we point out that in the case of $\mathbb{C}(z)$ there is an additional valuation "at infinity" corresponding to $\infty \in \mathbb{CP}^1$, and there is no such valuation for $\mathbb{Q}$.

4. Conclusion

Section 3 demonstrates the manifold applications of Theorem 1.1 and the tools used to prove it, especially to algebraic geometry and number theory. What is most apparently beautiful is the aforementioned equivalence of categories between

(1) compact Riemann surfaces and holomorphic maps between them;

(2) projective, nonsingular complex algebraic curves and nonconstant morphisms between them;

(3) finite field extensions of $\mathbb{C}(z)$.

Recall that we can move between these categories in the following ways: Given a Riemann surface, considering the meromorphic functions on the surface gives us a field. Likewise, given any algebraic curve, we can consider the *field of rational functions* on the curve. Starting with a field K, we saw in

Theorem 2.1 that we can construct a Riemann surface with K as its field of meromorphic functions; Chow's theorem then tells us that this Riemann surface is an algebraic curve. Alternatively, we saw in the proof of Theorem 2.1 that K corresponds to the compactification of some complex algebraic curve. What is most striking about this equivalence of categories is that it allows us to work seamlessly in the intersection of algebra, geometry, and analysis! For example, holomorphic maps between compact Riemann surfaces—analytic objects—are secretly just morphisms of algebraic curves, and these are secretly just inclusions of fields! Thus, this result allows us to turn analysis into geometry into algebra and vice versa. Results such as Theorem 3.3 are particularly useful because they allow us to turn the problem of computing the monodromy group of a holomorphic map into the problem of computing the Galois group of some field extension, which, in some cases, is simpler or at the very least more accessible.

To conclude, we present a true proof of Theorem 1.2 (the impetus for this project), i.e., one that does not rely on Theorem 1.1 (which we did not prove entirely).

Proof of Lüroth's Theorem. By assumption, K must have transcendence degree 1 over $\mathbb{C}$. Let X denote the compact Riemann surface from Theorem 2.1 associated to K so that the meromorphic functions on X are $k_X \cong K$. The existence of a meromorphic function $f : X \to \mathbb{CP}^1$ tells us that there is an extension of fields $K \cong k_X/\mathbb{C}(z)$ with $[K : \mathbb{C}(z)] = \deg(f)$. This forces $\deg(f) = 1$, implying that it is a biholomorphism from $X \to \mathbb{CP}^1$, as desired. ☐

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